Slack variety of a polytope and its applications

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Section 1

Polytopes and their realization spaces

Polytopes

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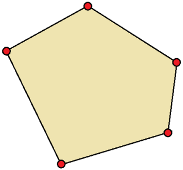

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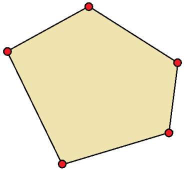

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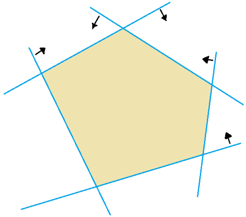

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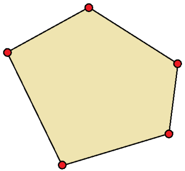

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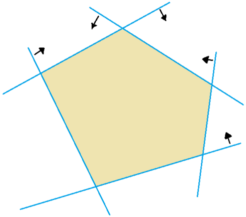

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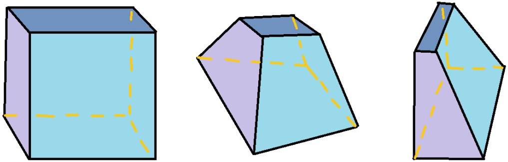
$\mathcal{H}$-representation

A face of P is its intersection with a supporting hyperplane, and the set of faces ordered by inclusion forms the face lattice of P

Combinatorial class of a polytope

We say that two polytopes are combinatorially equivalent if they have the same face lattice.



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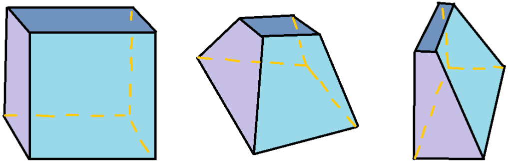

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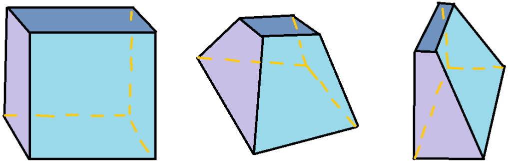

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Question: How do we make such an object concrete?

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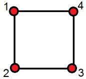

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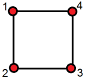

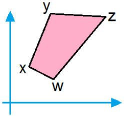

$$\mathcal{R}(P) = \left\{ \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \end{bmatrix} : \begin{array}{c} w, x, y, z \text{ are} \\ \text{vertices of a square} \end{array} \right\}$$

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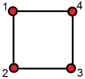

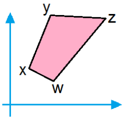

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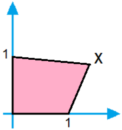
We can also mod out affine transformations by fixing an affine basis B.



$$\begin{aligned} \mathcal{R}(P,B) &= \left\{ \begin{bmatrix} 0 & 0 & 1 & x_1 \\ 1 & 0 & 0 & x_2 \end{bmatrix} : \begin{array}{c} e_1, 0, e_2, x \text{ are} \\ \text{vertices of a square} \end{array} \right\} \\ &= \{x \in \mathbb{R}^2 \ : \ x_1, x_2 \geq 0, x_1 + x_2 \geq 1\} \end{aligned}$$

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We will present an alternative construction for a model of the realization space that will be suitable to some different applications.

Section 2

Slack variety of a polytope

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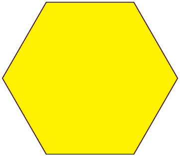

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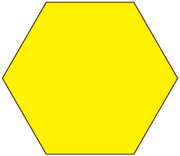

Its 6×6 slack matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

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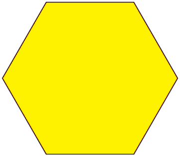

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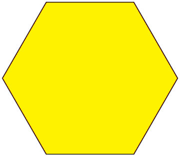

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- The slack matrix is defined only up to column scaling;
- The slack matrix can't see affine transformations;
  Moreover P is affinely equivalent to the convex hull of the rows of S_P.

Characterization of slack matrices

If P is a d-polytope with $\mathcal{V}$-representation $\{p_1, \dots, p_v\}$ and $\mathcal{H}$-representation $Ax \leq b$ then

$$S_P = \begin{bmatrix} b & -A \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_v \end{bmatrix}$$

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A nonnegative matrix S is the slack matrix of some realization of P if and only if

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There is a one-to-one correspondence between matrices with those properties (up to column scaling) and realizations of P (up to affine equivalence).

Projective equivalence

In general, we will be interested in modding out projective transformations.

$$Q \overset{p}{=} P \Leftrightarrow Q = \phi(P), \quad \phi(x) = \frac{Ax + b}{c^\intercal + d}, \quad \det \begin{bmatrix} A & b \\ c^\intercal x & d \end{bmatrix} \neq 0$$

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Slack matrices offer a natural way of quotient projective transformations.

Theorem (GPRT, 2017)

$Q \overset{p}{=} P \Leftrightarrow S_Q = D_v S_P D_f$ *for some positive diagonal matrices* D_v, D_f

Slack ideals

Slack ideal

Let P be a d-polytope and $S_P(x)$ a symbolic matrix with the same support as S_P. Then the slack ideal of P is

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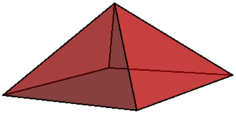

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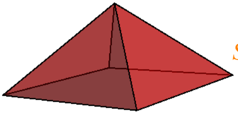

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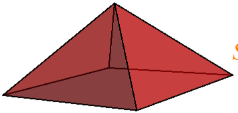

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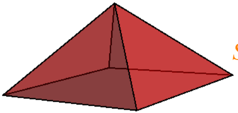

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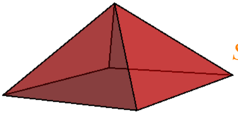

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Theorem (GMTW, 2017)

$\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0}) \xleftrightarrow{1:1}$ *classes of projectively equivalent polytopes of the same combinatorial type as P.*

Slack realization space

- $\mathcal{V}(I_P)$ is the slack variety of P.
- Positive part of slack variety: $\mathcal{V}_+(I_P) = \mathcal{V}(I_P) \cap \mathbb{R}^n_+$
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We call $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ the slack realization space of P.

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If for every facet k of P we pick a set I_k of $d-1$ spanning vertices we can define a matrix

$$(S(\tilde{x}))_{k,l} = \pm \tilde{x}_{(I_k,l)}$$

This is a slack matrix of P and its row space is $\bar{x}$.

Section 3

Applications

Application 1: Psd-minimality

A semidefinite representation of size k of a d-polytope P is a description

$$P = \left\{ x \in \mathbb{R}^d \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where A_i and B_i are $k \times k$ real symmetric matrices.

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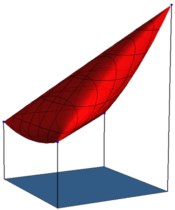

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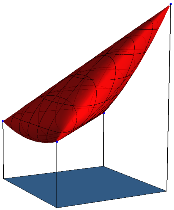

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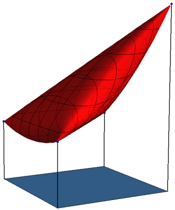

Optimizing over such sets is "easy": we want small representations.
Turns out the smallest possible size is $d+1$. When does that happen?

Application 1: Psd-minimality (part 2)

Theorem (GRT 2013; GGS 2016)

- *A polytope P is psd-minimal* $\Leftrightarrow \exists S_p(y) \in \mathcal{V}_{\mathbb{R}}(I_P)$ *such that* $S_P = S_P(y^2)$.
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- In $\mathbb{R}^3$ who knows?...

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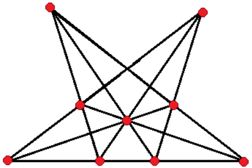

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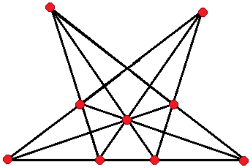

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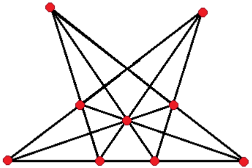

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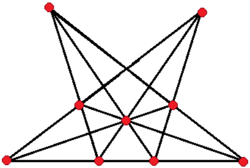

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This can be extended to the ideal of the Perles polytope (d=8, v=12, f=34)
It is not rational but also its slack ideal is not prime.

Application 3: Realizability

Steinitz problem *Check whether an abstract polytopal complex is the boundary of an actual polytope.*

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[Altshuler, Steinberg, 1985]: 4-polytopes and 3-spheres with 8 vertices.

The smallest non-polytopal 3-sphere has vertex-facet non-incidence matrix

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In this case, $I_P = \langle 1 \rangle \Rightarrow$ no rank 5 matrix with this support $\Rightarrow$ no polytope.

Section 4

One more application

Dimension of the realization space

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Can we compute the dimension of $\mathcal{V}(I_P)$?

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However

3. Maybe we can use the structure of the variety to do enough?

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Let us go to a related more basic problem:

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In general, Dykstra's alternate projection algorithm will applied to $\bar{S} = S_P$+noise will converge to the projection of $\bar{S}$ in $\mathcal{V}(I_P)$.

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This is not a full answer to the question, but might be enough.

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In particular the average distance squared should converge to the dimension!

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Theorem (Padrol-Sanyal 2016)

Let $I_{n,k}$ be the slack ideal of $H_{n,k}$. For $k \geq 2$, we have

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n \ k	2	3	4
4	**16**/16.0		
5	**25**/25.0		
6	**36**/36.0	**41**/41.0	
7	**49**/49.0	**63**/63.0	
8	**64**/64.1	**92**/91.8	**106**/105.9
9	**81**/81.0	**129**/129.0	**171**/171.0

Lets try it out some more

Given a poset P with base elements $\{1, \ldots, n\}$ its order polytope is

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We tried many three dimensional polytopes, projectively unique polytopes and pretty much everything we could got our hands on. All worked.

Conclusion

There are many more questions, and a more algebraic perspective.

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For further reading:

- arXiv:1708.04739 - *The Slack Realization Space of a Polytope*
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Thank you